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A SECOND INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS*

BY CARL EBEN STROMQUIST

THE simplest problem in the calculus of variations is the following: Given an integral

$$(1) \qquad I = \int_{x_1}^{x_2} g(x, y, p)\, dx, \qquad \text{where } p = \frac{dy}{dx},$$

to find a curve $y = y(x)$ which joins two given fixed points and renders this integral a minimum. A first necessary condition for a minimum is that $y = y(x)$ must satisfy the Euler differential equation

$$(2) \qquad y'' g_{pp} + p g_{py} + g_{px} - g_y = 0,$$

where $y'' = d^2y/dx^2$ and the subscripts denote partial differentiation. The two-parameter family of curves

$$(3) \qquad y = y(x, \alpha, \beta)$$

which the differential equation (2) yields is called the family of extremals of the integral I.

The so-called inverse problem of the calculus of variations arises when a two-parameter family of curves

$$y = y(x, \alpha, \beta),$$

and hence their differential equation

$$(4) \qquad y'' = \phi(x, y, p),$$

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* Read before the American Mathematical Society, April 27, 1907.

are given and one seeks to determine the integrand $g(x, y, p)$ so that the given family of curves (3) will be extremals for the integral

$$I=\int_{x_1}^{x_2} g(x, y, p)\,dx.$$

The inverse problem in this general form has been discussed by Darboux.[*] If the extremals are to be the solutions of equation (4), the value of y'' substituted in (2) must reduce this equation to an identity in x, y, p and hence the equation

$$\phi(x, y, p) g_{pp} + p g_{py} + g_{px} - g_y = 0$$

will be a differential equation for g. As there are an infinite number of values of g which will satisfy this differential equation, the inverse problem has in general an infinite number of solutions. Completely to determine g some further conditions must therefore be imposed. It has been shown[†] that in the cases where straight lines or circles with centers on the x-axis are extremals, if the further condition be imposed that extremals are perpendicular to their transversals, then the function g is completely determined aside from a constant factor. The condition of transversality is a relation between the variables x, y, p, P, where p is the slope of the extremal at the point (x, y), and P the slope of the curve cutting the extremal transversally at that point. The equation expressing this relation is in fact[‡]

$$g(x, y, p) + (P-p) g_p(x, y, p) = 0, \tag{5}$$

which when solved for P takes the form[§]

$$P = \theta(x, y, p). \tag{6}$$

We have seen above that when the integral (1) is given the extremals (3) can always be determined, or inversely when the family of curves (3) is given then an integral with these as extremals can always be found.

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[*] Darboux, *Lecons*, vol. 3, p. 53. See also Bolza, *Lectures on the Calculus of Variations*, p. 30.

[†] Stromquist, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 181.

[‡] Bolza, loc. cit., pp. 36, 106, 172. Geometrically stated, two transversals to the same set of extremals are two curves which intercept on the extremals arcs along which the integral under consideration has a constant value.

[§] It is supposed in what follows that the function θ and its first partial derivatives exist and are continuous in x, y, p and that $p - \theta \neq 0$.

A second inverse problem in connection with the integral (1) is the following: If a relation of the form (6) is given, i. e., the relation between the direction whose slope is p and the direction transverse to it at each point of the plane, then will it always be possible to determine an integral (1) whose extremals and transversals satisfy the given relation? It will be shown in §1 that a function g giving rise to the relation (6) can always be determined, and the further conditions which insure that it is uniquely determined will be derived. In §2 the results of §1 will be applied to some examples.

1. Determination of the integral corresponding to a given relation of transversality. If the integral (1) is to have the equation (6) as its relation of transversality then it is evident from (5) that the integrand g must be a solution of the equation

$$g(x, y, p) + \{\theta(x, y, p) - p\} g_p(x, y, p) = 0$$

for all values of x, y, p. This is a linear differential equation for g which is easily integrated. The solution has the form

$$g = \psi(x, y) e^{A}, \tag{7}$$

where

$$A = \int_0^p \frac{dp}{p - \theta(x, y, p)},$$

and $\psi(x, y)$ is an arbitrary function of x and y. We thus have as a first result: *If an integral*

$$I = \int_{x_1}^{x_2} g(x, y, p)\, dx$$

has the equation

$$P = \theta(x, y, p)$$

as its relation of transversality, then the integrand g must be of the form

$$g = \psi(x, y) e^{A},$$

where

$$A = \int_0^p \frac{dp}{p - \theta(x, y, p)},$$

and $\psi(x, y)$ is an arbitrary function of x and y.

In order to determine the arbitrary function $\psi(x, y)$ which enters into the expression for g, suppose that it be further required that a one-parameter family of curves

$$y = y(x, \gamma) \tag{8}$$

be given which shall be extremals for the integral I. It is well known that if along an arc of one of these extremals the derivative $\partial y/\partial\gamma$ is different from zero then a neighborhood of that arc can be found which is simply covered by the extremals (8) and which constitutes a so-called field.* Let us suppose that such a field has been found. Then at each point (x, y) of the field the equation (8) has a solution

$$\gamma = \gamma(x, y),$$

which gives the value of γ for the extremal of the family which passes through that point. The slope of the extremal at the point (x, y) has the value

$$p = \frac{dy(x, \gamma)}{dx} = \frac{dy(x, \gamma(x, y))}{dx} = p(x, y), \tag{9}$$

and the second derivative of y is

$$y'' = p_x + pp_y. \tag{10}$$

Hence, if the given relation (6) is to exist between the directions of the extremals and transversals and if further $y = y(x, \gamma)$ is to be a one-parameter family of extremals for the integral (1), then the Euler differential equation (2) must be satisfied identically when the value of g given by (7) and the value of y'' given by (10) are substituted. The Euler equation then becomes

$$\frac{(p_x + pp_y)\theta_p + \theta_x + p\theta_y}{p - \theta} + \int_0^p \frac{\theta_x dp}{(p - \theta)^2} + \theta \int_0^p \frac{\theta_y dp}{(p - \theta)^2} + \frac{1}{\psi}\Big(\psi_x + \theta\psi_y\Big) = 0, \tag*{11)}$$

where the value $p(x, y)$ from equation (9) is to be substituted for p. Equation (11) is a partial differential equation of the first order for the function ψ. It can be written in the form

$$\psi_x + \theta\psi_y + K(x, y)\psi = 0, \tag{12}$$

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* Bolza, loc. cit., §19; Osgood. Annals of Mathematics, ser. 2, vol. 2, p. 112; Goursat, *Cours d'Analyse*, vol. 2, §451. See also these references for conditions of continuity etc. that must be satisfied by $y(x, \gamma)$.

where

$$K(x, y) = \frac{(p_x + pp_y)\theta_p + \theta_x + p\theta_y}{p - \theta} + \int_0^p \frac{\theta_x dp}{(p - \theta)^2} + \theta \int_0^p \frac{\theta_y dp}{(p - \theta)^2}.$$

It is evident that this equation still does not uniquely determine the function ψ.

In order to integrate equation (12) we have the auxiliary set of equations*

$$\frac{dx}{1} = \frac{dy}{\theta} = \frac{d\psi}{-K(x, y)\psi}. \tag{13}$$

Let the solution of

$$dx = \frac{dy}{\theta}$$

be taken in the form

$$u(x, y) = c. \tag{14}$$

If this is solved for y in terms of x and c giving $y = \eta(x, c)$ then the solution of

$$\frac{d\psi}{-K(x, \eta)\psi} = \frac{dx}{1}$$

will be of the form

$$\psi \cdot v(x, y) = c_1, \tag{14'}$$

where v is a particular solution of the equation

$$\frac{dv}{dx} - K(x, \eta)v = 0,$$

and is found by solving this equation regarding c as a constant and afterwards replacing c by the function $u(x, y)$. Then the most general solution of equation (12) is

$$\psi \cdot v(x, y) = W(u), \tag{15}$$

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* Darboux, loc. cit., vol. 3, p. 55; Jordan, *Cours d'Analyse*, vol. 3, §242.

where W is an arbitrary function of u. Equation (7) now reduces to

$$(16) \qquad g(x, y, p) = \frac{W(u)}{v(x, y)} \cdot e^{A},$$

where
$$A = \int_0^p \frac{dp}{p - \theta(x, y, p)}.$$

Here u and v are known functions of x and y but W is an arbitrary function of u.

It is to be observed that any curve $u = c$ is a transversal in the field of extremals (8) since it is the solution of $dy/dx = \theta$. If the value of I is given along any curve of the field which is not one of these transversals or tangent to one of them, then the function $W(u)$, and consequently $g(x, y, p)$, is uniquely determined. For along such a curve, $y = y(x)$, the derivative

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

is different from zero so that u is an always increasing (or always decreasing) function of x along the curve, and conversely x is an always increasing (or always decreasing) function $x = x(u)$ of u. Since now I is an assigned function of x along the curve $y = y(x)$ it follows that $dI/dx = g$ in equation (16) is uniquely determined. Hence if we substitute in the right hand member of equation (16) $y = y(x)$ and $p = y'(x)$, and then replace x by the function $x = x(u)$, the result will be an identity in u which uniquely determines the function W. We have then attained the following result:

If besides the relation of transversality (6) *it is also prescribed that the integral I shall have a one-parameter family of curves $y = y(x, \gamma)$* (8) *arbitrarily chosen, as extremals, then the function $\psi(x, y)$ in equation* (7) *must satisfy a partial differential equation* (12). *The value of $\psi(x, y)$ is completely determined if in a field composed of extremals* (8) *the value of the integral I as a function of x is given along any curve which is not tangent at any point to a transversal of the field.*

2. Examples. In this section the above results will be applied to some examples.

Example 1. Consider the case where the extremals and transversals are perpendicular, i. e., where

$$P = \theta = -\frac{1}{p}. \tag{17}$$

It follows readily from equation (7) that then g must have the form

$$g = \psi(x, y)\sqrt{1 + p^2}, \tag{18}$$

and equation (11), from which ψ is to be found, becomes

$$(p_x + pp_y)\psi + (1 + p^2)(p\psi_x - \psi_y) = 0. \tag{19}$$

Example 1*a*. Suppose it be further required that the one-parameter family of straight lines through the fixed point $P_0(x_0, y_0)$, viz.

$$y - y_0 = m(x - x_0),$$

be extremals. Since then

$$p = \frac{dy}{dx} = m = \frac{y - y_0}{x - x_0},$$

it follows that

$$p_x + pp_y = 0,$$

and hence equation (19) becomes

$$(y - y_0)\psi_x - (x - x_0)\psi_y = 0. \tag{20}$$

The auxiliary set of equations to be integrated in this case is

$$\frac{dx}{1} = \frac{dy}{-\dfrac{x - x_0}{y - y_0}} = \frac{d\psi}{0},$$

from which are obtained the following results—corresponding to equations (14) and (14′):—

$$u = (x - x_0)^2 + (y - y_0)^2 = c^2, \tag{21}$$

$$\psi = c_1;$$

hence the general solution of equation (19)—corresponding to equation (15)—is

$$\psi = W\{(x - x_0)^2 + (y - y_0)^2\}.$$

In this case, therefore, g must have the form

$$g(x, y, p) = W(u)\sqrt{1 + p^2}.$$

The transversals, as is seen from equation (21), are a family of circles with centres at the given point (x_0, y_0). Suppose that along some curve L which passes through (x_0, y_0) and is not tangent at any point to one of these transversals the value of the integral

$$\int_{x_1}^{x} g\,dx = \int_{x_1}^{x} W(u)\sqrt{1 + p^2}\,dx$$

be equal to the integral $\int_{x_1}^{x} \sqrt{1 + p^2}\,dx$. It then follows by differentiating for x that for all values of u along the given curve, $W(u) = 1$. Hence g is completely determined and must be of the form $g = \sqrt{1 + p^2}$ at least for all points in a circle about (x_0, y_0) as a centre and with the largest value of u on L as a radius.

These results have an interesting application in the theory of plane geometry. We may assume that the elements of our geometry are the number pairs (x, y) and may then define length * along any curve to be given by an integral of the form

$$I = \int g(x, y, p)\,dx. \tag{22}$$

The nature of the geometry is then determined by the character of the function g and in the special case when g has the form $\sqrt{1 + p^2}$ we get the ordinary theory of Euclidean geometry. The extremals for the integral (22) play the role of the straight lines in the ordinary geometry and the circles about a point P_0 correspond to the transversals to the one-parameter family of extremals through P_0. The condition of transversality (6) is then a relation between

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* This is practically the standpoint of Hamel in his dissertation, Göttingen, 1901. See also *Transactions Amer. Math. Society*, vol. 7 (1906), p. 175, third foot-note.

the directions of a shortest line and a circle. With these geometrical notions, the above results can be stated as follows:

If the length of an arc of a curve from (x_1, y_1) *to* (x, y) *be defined as the value of the integral* $I = \int_{x_1}^{x} g(x, y, p)\, dx$, *and if circles are required to be perpendicular to their radii (i. e., transversals perpendicular to extremals), then* g *must be of the form* $g = \psi(x, y)\sqrt{1 + p^2}$. *If it be further true that there is a single point* P_0 *such that the one-parameter family of shortest lines through* P_0 *are all straight lines and if the integral* I *takes the same values as the Euclidean length integral along a single curve through the point* P_0 *and not tangent to any of the circles about* P_0, *then it follows that* I *must be everywhere equal to* $\int_{x_1}^{x} \sqrt{1 + p^2}\, dx$.

Example 1*b*. If it be required that the one-parameter family of circles with centres on the x-axis and passing through the point (x_0, y_0), viz.

$$x^2 - x_0^2 + y^2 - y_0^2 - 2a(x - x_0) = 0, \tag{23}$$

be extremals, then equation (19) reduces to

$$\{(x - x_0)^2 - (y^2 - y_0^2)\}\psi_x + 2y(x - x_0)\psi_y + 2(x - x_0)\psi = 0. \tag{24}$$

The equations corresponding to equations (14) and (14′) are in this case

$$u = \frac{(x - x_0)^2 + y^2 + y_0^2}{y} = c,$$

$$y \cdot \psi = c_1.$$

Hence the most general solution of equation (24) is

$$\psi = \frac{1}{y} W\left\{\frac{(x - x_0)^2 + y^2 + y_0^2}{y}\right\},$$

and therefore g must have the form

$$g = \frac{1}{y} W(u)\sqrt{1 + p^2}.$$

In a manner analogous to that of the preceding example it can now be shown that, if we further require that along a single curve through the point (x_0, y_0) and not tangent to a transversal the value of

$$\int_{x_1}^{x} g\,dx = \int_{x_1}^{x} \frac{1}{y} W(u)\sqrt{1+p^2}\,dx$$

be equal to

$$\int_{x_1}^{x} \frac{1}{y}\sqrt{1+p^2}\,dx,$$

then g is completely determined and the integral I must have the form

$$\int_{x_1}^{x} g\,dx = \int_{x_1}^{x} \frac{1}{y}\sqrt{1+p^2}\,dx.$$

This is the form of the length integral obtained by a conformal transformation of the pseudo-sphere upon the plane.* We have therefore obtained the following result:

If it be required that circles are perpendicular to their radii (i. e., transversals perpendicular to extremals) and also that a one-parameter family of extremals through P_0 be the circles through this point and with centres on the x-axis, then it follows that the integral I is uniquely determined and has the form

$$I = \int_{x_1}^{x} \frac{1}{y}\sqrt{1+p^2}\,dx,$$

i. e., has the form of the length integral obtained by a conformal transformation of the pseudo-sphere upon the plane, provided that the integral has this value along a single curve through P_0 which is not tangent to any of the circles (transversals) about P_0.

Example 2. The case where a linear fractional relation,

$$P = \theta(x, y, p) = \frac{ap+b}{cp+d}, \tag{25}$$

exists between P and p naturally suggests itself. Here a, b, c, d are functions of x and y. It follows from equation (5) that g in this case must be a solution of the linear differential equation

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* See *Transactions*, loc. cit., p. 181.

$$\frac{dg}{g} = \frac{(cp + d)\,dp}{cp^2 + (d - a)p - b}$$

for all values of x and y. This is readily integrated and the solution has the form

$$g(x, y, p) = \psi(x, y)\sqrt{cp^2 + (d - a)p - b} \cdot e^{\frac{a+d}{2}H}, \tag{26}$$

where

$$H = \frac{2}{\sqrt{-q}}\,tan^{-1}\frac{2cp + (d - a)}{\sqrt{-q}}$$

if $cp^2 + (d - a)p - b$ is a definite quadratic form, or

$$H = \frac{1}{\sqrt{q}}\log\frac{2cp + d - a - \sqrt{q}}{2cp + d - a + \sqrt{q}}$$

if $cp^2 + (d - a)p - b$ is an indefinite quadratic form. Here $q = (d - a)^2 + 4bc$. If $cp^2 + (d - a)p - b$ is a semi-definite quadratic form, then

$$g = \psi(x, y)(2cp + d - a)e^{-\frac{d+a}{2cp+d-a}}.$$

Example 3. It is well known from the theory of surfaces* that if the line element of a surface be of the form

$$g\,dx = \sqrt{E + 2Fp + Gp^2}\,dx,$$

where x and y are the arguments of E, F, G in place of the ordinary u and v, then the condition that a family of curves

$$\phi_2(x, y) = \text{constant}$$

shall be the orthogonal trajectories of a given family of curves

$$\phi_1(x, y) = \text{constant}$$

is that

$$P = \frac{-Fp - E}{Gp + F},$$

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* Bianchi, *Vorlesungen über Differentialgeometrie*, p. 66.

where

$$p = -\frac{\dfrac{\partial\phi_1}{\partial x}}{\dfrac{\partial\phi_1}{\partial y}} \qquad \text{and} \qquad P = -\frac{\dfrac{\partial\phi_2}{\partial x}}{\dfrac{\partial\phi_2}{\partial y}}.$$

From equation (26) of the preceding example it follows at once that *if it be required that the condition of orthogonality of two curves on a surface be*

$$P = \frac{-Fp - E}{Gp + F},$$

where E, F, G are functions of the parameters x and y of the surface, then the line element of the surface must have the form

$$g\,dx = \psi(x, y)\sqrt{E + 2Fp + Gp^2}\,dx.$$

Example 4. As a further special case of example 2, it follows readily that if extremals and transversals cut at a constant angle $\beta = \tan^{-1} k$, i. e., if

$$\theta = \frac{p - k}{kp + 1},$$

then the integrand g must have the form

$$g = \psi(x, y)e^{\frac{1}{k}\tan^{-1}p}\sqrt{1 + p^2}.$$

In this case if a one-parameter family of straight lines through the origin, $y = mx$, be extremals then it follows that the transversals will be the spirals

$$\sqrt{x^2 + y^2} \cdot e^{\frac{1}{k}\tan^{-1}\frac{y}{x}} = c \qquad \text{or} \qquad \rho = ce^{-\frac{\theta}{k}}.$$

If the circles $x^2 + y^2 - 2ax = 0$ are extremals the transversals are the circles

$$(x - c)^2 + (y - ck)^2 = c^2(1 + k^2).$$

If the circles $x^2 + y^2 = a^2$ are extremals the transversals are the spirals

$$\sqrt{x^2 + y^2}\, e^{-k\tan^{-1}\frac{y}{x}} = c \qquad \text{or} \qquad \rho = ce^{k\theta}.$$

PRINCETON, N. J.,
JUNE, 1907.